

Analysis and estimation of the threshold for a microwave “pellicle mirror” parametric oscillator, via energy conservation

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Abstract: An experiment is proposed to observe the dynamical Casimir effect by means of two tandem, high Q , superconducting microwave cavities, which are separated from each other by only a very thin wall consisting of a flexible superconducting membrane that can be driven into motion by means of resonant “pump” microwaves injected into the left cavity. Degenerate “signal” and “idler” microwave signals can then be generated by the exponential amplification of vacuum fluctuations in the initially empty right cavity, above a certain threshold. The purpose of this paper is calculate the threshold for this novel kind of opto-mechanical parametric oscillation, using energy considerations.

In order to be able to numerically estimate the threshold for the “pellicle mirror” parametric oscillator, let us start from the LC equivalent circuit for the TM 010 mode of a cylindrical pillbox cavity that is illustrated in Figure 1, parts (a) and (b). Charges of the pillbox cavity in (a) accumulating on opposite sides of the inner end faces, which produce electric field lines (in green) that run straight across from the left inner end face to the right inner end face, can be simulated by charges accumulating on the opposite plates of the capacitor C in the LC circuit in (b), which produce electric field lines (in green) that run straight across from the left plate to the right plate. In other words, we shall model, in a crude first approximation, the *nonuniform* electric field inside the TM 010 mode of the pillbox cavity by means of the *uniform* electric field inside the capacitor C of the LC circuit, in order to be able to perform a first, rough estimate of the threshold.

We shall calculate the threshold using a classical field analysis. Suppose that the left end face of the pillbox cavity were to be kept fixed, but that the right

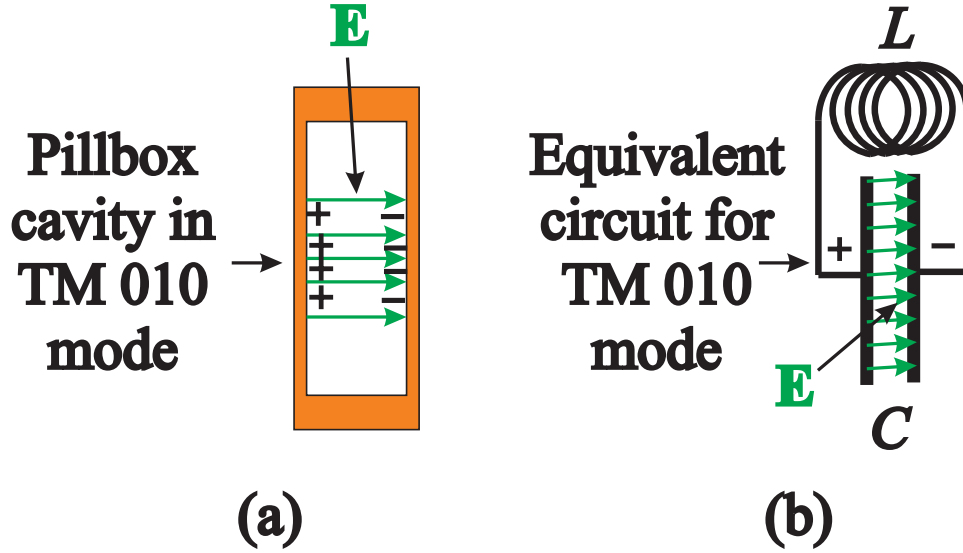


Figure 1: (a) The microwave TM 010 mode of a cylindrical “pillbox cavity” pictured at the instant when the electric field lines \mathbf{E} reach a maximum. (b) The equivalent LC circuit for this cavity mode, also pictured at the instant when the electric field lines \mathbf{E} reach a maximum inside the capacitor C .

end face were to be replaced by a thin, flexible, conducting “pellicle mirror” that could be driven into mechanical motion. The analogous situation in the equivalent LC circuit would be for the left plate of the capacitor C to be kept fixed, but the right plate to be mechanically driven into motion by some external agency, as illustrated in Figure 2.

For now, let us assume that this motion is so sudden that the charges $+Q$ and $-Q$ on the left and right plates, respectively, of the capacitor C , at the moment of maximum charge accumulation in the LC circuit, cannot instantaneously change their values during the sudden displacement Δx . This is because the inductor L of the LC circuit in Figure 1, part (b), forbids any instantaneous change in the current flowing through it at the instant of maximum charge accumulation pictured in Figure 2.

The work that is done by the external agency (the “pump”) on the moving plate of the capacitor that produces a sudden displacement of an amount Δx towards the right, as pictured in Figure 2, is given by

$$\Delta W = \left(\frac{1}{2} \epsilon_0 E^2 \right) \cdot A \Delta x \quad (1)$$

where

$$u_E = \frac{1}{2} \epsilon_0 E^2 \quad (2)$$

is the energy density of the electric field inside the capacitor, and where the

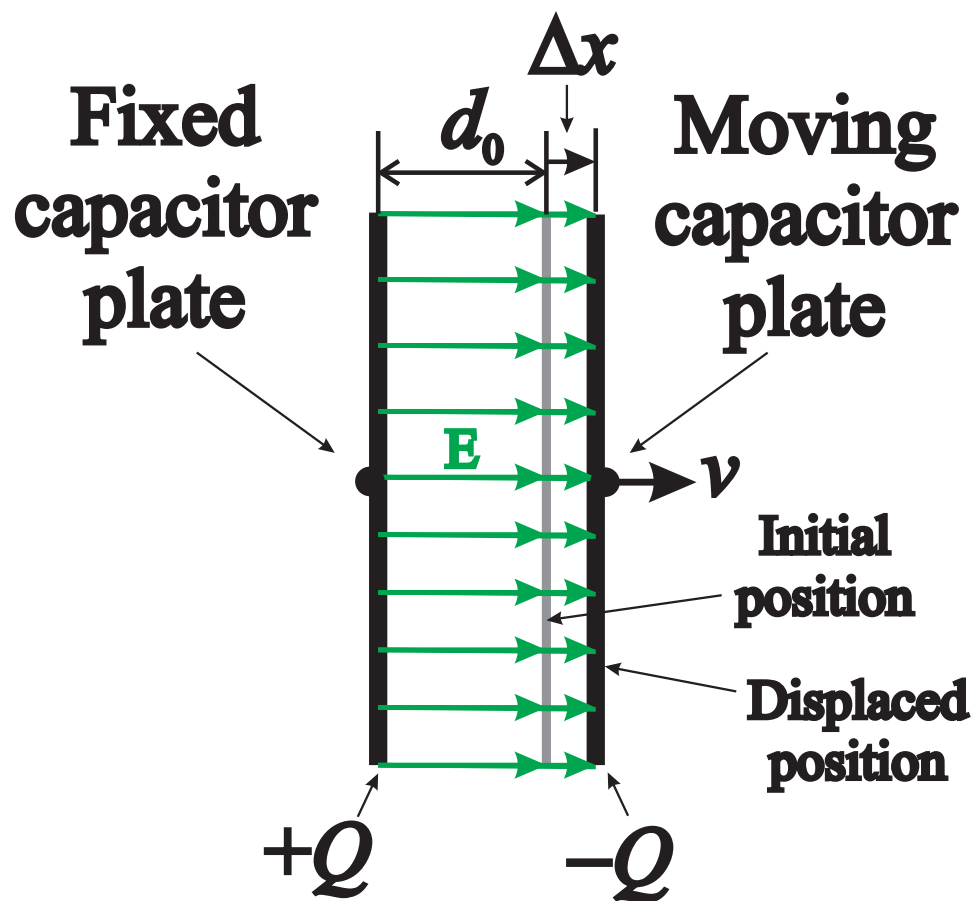


Figure 2: Magnified view of the capacitor C in which the left plate is fixed, but the right plate is displaced towards the right due to the action of some “pump.”

change of the volume ΔV inside the capacitor due to the sudden displacement Δx , is given by

$$\Delta V = A\Delta x \quad (3)$$

where A is the area of the capacitor plate. It is to be stressed that the sudden displacement Δx is caused by some *external* agency.

The mechanical work ΔW in (1) done by this external agency on the fields inside the LC circuit can be rewritten in the following form

$$\Delta W = P\Delta V \quad (4)$$

where the pressure P on the plate of the capacitor is given by

$$P = \frac{1}{2}\varepsilon_0 E^2 \quad (5)$$

which is equal to the instantaneous energy density u_E given by (2) [1]. Therefore the work in (4) can be viewed as the *negative* of the usual thermodynamic “ PdV ” work done *by* a closed “system” (here, the electric fields inside the capacitor) [2][3] *on* the “environment” (here, the external agency). In this way, one can view the moving “pellicle mirror,” i.e., the moving end wall of the closed pillbox cavity, or the moving capacitor plate in the equivalent LC circuit, as if it were a moving piston acting on a closed thermodynamic system, namely, the vacuum fluctuations inside the cavity, or equivalently, inside the LC circuit.

The pressure $P = \frac{1}{2}\varepsilon_0 E^2$ in (5) can also be viewed as being one component of the Maxwell stress tensor [4]

$$T_{ij} = \varepsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2} \delta_{i,j} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \right] \quad (6)$$

namely, the component with spatial indices $i = j = 3$, i.e., the tensor product of the *longitudinal* electric field at the surface of the plate with itself, at the instant when no magnetic fields are present.

Note that there are no electrical charges within the volume $\Delta V = A\Delta x$ which is being swept out by the motion of the right plate in Figure 2. Therefore, using Maxwell’s first equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = 0 \quad (7)$$

and using a Gaussian pillbox argument, we conclude that the lines of electric field \mathbf{E} , which originated from the charge $+Q$ on the left plate, must continue straight across this newly created volume, and terminate on the charges $-Q$ on the right plate at its new, displaced position, as indicated in Figure 2. This implies that there must exist a process of “creation out of nothing” of new electric field lines of E inside the new volume ΔV , and therefore that, during this process, there must also exist an accompanying creation of an extra amount of electromagnetic energy in the LC circuit,

$$\Delta U_E = \left(\frac{1}{2} \varepsilon_0 E^2 \right) \Delta V \quad (8)$$

seemingly out of nothing, due to the sudden displacement of this plate to the right by the amount Δx .

However, energy must be conserved. Therefore the meaning of (1) is it is that the amount of *mechanical* work ΔW done on the plate must be exactly equal to the amount of *electrical* energy ΔU_E in (8), which is being created within the LC system by the external “pump” agency, upon the sudden displacement Δx of the moving plate towards the right.

Now if energy were to be continually supplied to the LC circuit by a periodic sequence of displacements $\Delta x(t)$ to the mirror by some continuous-wave external pumping agency at a correct frequency, exponential amplification of some seed waveform within the circuit could result, if a certain synchronization condition is met. Here we shall assume that the LC circuit is on resonance at the *same* frequency for both signal and idler waves, i.e., we are only considering a *degenerate* parametric amplifier. Then the synchronization condition follows from the fact that in the expression for the electromagnetic energy $\Delta U_E = (\frac{1}{2}\epsilon_0 E^2) \Delta V$ inside the resonator, the factor of E^2 implies that there would be a second harmonic term in the pressure P in (5). Hence if we choose the origin in time such that the waveform for the electric field inside the capacitor is given by

$$E = E_0 \cos \omega t \quad (9)$$

then it follows that the square of the electric field is given by

$$E^2 = E_0^2 \cos^2 \omega t = E_0^2 \cdot \frac{1}{2} (1 + \cos 2\omega t) \quad (10)$$

where ω is the resonance frequency of the LC circuit, and therefore that the pressure on the plate in (5)

$$P = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{4} \epsilon_0 E_0^2 (1 + \cos 2\omega t) \quad (11)$$

has a second harmonic term that could then be made synchronous with the second harmonic pumping term in the mechanical work ΔW in (1). We can do so by choosing the periodic displacement waveform $\Delta x(t)$ driven by the pump source to coincide in frequency with the second harmonic term in the pressure on the plate in (11).

This motivates us to postulate that the external pump agency constrains the right plate of the capacitor C in Figure 2 to move so that its velocity moves at the second harmonic frequency according to the expression

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = v_{2\omega} \cos 2\omega t \quad (12)$$

where $v_{2\omega}$ is a fixed drive velocity amplitude that is determined by the “pump.” Then there arises from the mechanical work in (1) an expression for the instan-

taneous mechanical power, which contains a time-varying factor

$$\begin{aligned}
\lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \left(\frac{1}{2} \varepsilon_0 E^2(t) \right) A \frac{\Delta x(t)}{\Delta t} = \left(\frac{1}{2} \varepsilon_0 E^2(t) \right) A v(t) \\
&= \left(\frac{1}{2} \varepsilon_0 E_0^2 \right) \cos^2 \omega t \cdot A v_{2\omega} \cos 2\omega t \\
&= \left(\frac{1}{4} \varepsilon_0 E_0^2 \right) (1 + \cos 2\omega t) \cdot A v_{2\omega} \cos 2\omega t \\
&= \left(\frac{1}{4} \varepsilon_0 E_0^2 \right) A v_{2\omega} (\cos 2\omega t + \cos^2 2\omega t)
\end{aligned} \tag{13}$$

which has a nonzero time average, since the time average of the last term in (13) gives

$$\langle \cos^2 2\omega t \rangle = \frac{1}{2} \tag{14}$$

where the angular brackets denote an average over time.

It therefore follows that the time-averaged power transfer from the external pump source into the LC circuit is

$$\left\langle \frac{dW}{dt} \right\rangle = \left(\frac{1}{8} \varepsilon_0 E_0^2 \right) A v_{2\omega} \tag{15}$$

Since the time-averaged energy density stored inside the capacitor is

$$\langle u_E \rangle = \frac{1}{2} \varepsilon_0 E_0^2 \langle \cos^2 \omega t \rangle = \frac{1}{4} \varepsilon_0 E_0^2 \tag{16}$$

we can rewrite (15) as follows:

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} \langle u_E \rangle A v_{2\omega} \tag{17}$$

In other words, energy is being continuously transferred from the external pump source into the LC circuit, with an average rate of transfer given by

$$\left\langle \frac{dW}{dt} \right\rangle = \left\langle \left(\frac{1}{2} \varepsilon_0 E^2 \right) A \frac{dx}{dt} \right\rangle = A \left\langle u_E \frac{dx}{dt} \right\rangle = A \langle u_E v \rangle = \langle F \cdot v \rangle \tag{18}$$

where

$$v = \frac{dx}{dt} = v_{2\omega} \cos 2\omega t \tag{19}$$

is the instantaneous velocity of the moving plate fixed by the external “pump,” and where $F(t) = P(t) A$ is the instantaneous force on this plate arising from the instantaneous pressure

$$P(t) = \frac{1}{2} \varepsilon_0 E^2(t) = \frac{1}{4} \varepsilon_0 E_0^2 (1 + \cos 2\omega t) \tag{20}$$

arising from the electric field inside the capacitor.

In the absence of any dissipation in the LC circuit, it follows from energy conservation that

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{d\langle U_E \rangle}{dt} + \frac{d\langle U_B \rangle}{dt} \quad (21)$$

Now it can be shown that the time-averaged energy stored in capacitor C of the LC circuit is equal to that stored in the inductor L , i.e.,

$$\langle U_E \rangle = \langle U_B \rangle \quad (22)$$

Therefore from (21), (22), and (17), it follows that

$$2 \frac{d\langle U_E \rangle}{dt} = \left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} \langle u_E \rangle A v_{2\omega} = +2\kappa \langle U_E \rangle \quad (23)$$

for some constant κ to be determined below. The ODE in (23) implies that there will be a continuous energy transfer from the pump source that will result in an *exponential* growth of the time-averaged stored energy $\langle U_E \rangle$ stored in the capacitor (and thus of the total energy in the LC circuit).

To find the proportionality constant κ , we note that the relationship between the total energy U_E stored in the capacitor and the energy *density* u_E stored in the capacitor, is given by

$$\langle U_E \rangle = \langle u_E \rangle A x_0 = \langle u_E \rangle V_0$$

where the equilibrium volume V_0 of the capacitor is given by

$$V_0 = A d_0 \quad (24)$$

where d_0 is the equilibrium spacing between the capacitor plates (see Figure 2). Therefore the ODE for $\langle U_E \rangle$ in (23) becomes

$$2 A x_0 \frac{d\langle u_E \rangle}{dt} = \frac{1}{2} \langle u_E \rangle A v_{2\omega} = +2\kappa \langle u_E \rangle A d_0 \quad (25)$$

Dividing by $2V_0 = 2Ad_0$, we obtain an ODE for $\langle u_E \rangle$

$$\frac{d\langle u_E \rangle}{dt} = \frac{1}{4} \langle u_E \rangle \frac{v_{2\omega}}{d_0} = +\kappa \langle u_E \rangle \quad (26)$$

Therefore we conclude that the exponential gain coefficient κ for the energy density in (26) is given by

$$\kappa = \frac{1}{4} \frac{v_{2\omega}}{d_0} = \frac{1}{4} \frac{v_{\text{pump}}}{d_0} \quad (27)$$

where

$$v_{2\omega} = v_{\text{pump}} \quad (28)$$

is the amplitude of the second-harmonic component of the instantaneous velocity (19) in the mechanical motion driven by a fixed, *undepleted* pump waveform that is determined by some external source.

Next, let us introduce a phenomenological loss coefficient γ into the ODE for $\langle u_E \rangle$ in (26) as follows:

$$\frac{d\langle u_E \rangle}{dt} = +\kappa \langle u_E \rangle - \gamma \langle u_E \rangle \quad (29)$$

If there is no gain (i.e., if $\kappa = 0$), then (29) becomes

$$\frac{d\langle u_E \rangle}{dt} = -\gamma \langle u_E \rangle = -\frac{1}{\tau} \langle u_E \rangle \quad (30)$$

where the “cavity ring-down time” τ is given by

$$\tau = \frac{1}{\gamma} = \frac{Q}{\omega} \quad (31)$$

where the quality factor of the cavity Q is defined through

$$Q = \omega\tau \quad (32)$$

The threshold for parametric oscillation, like that in a laser, occurs when the gain equals the loss, i.e., when

$$\kappa = \gamma \quad (33)$$

From (33), (31), and (27), we conclude that the threshold condition for the pump to be able to generate degenerate parametric oscillation, based on the moving capacitor plate model of Figure 2 for the LC equivalent circuit for the “pillbox cavity” of Figure 1, is given by

$$\kappa_{\text{threshold}} = \frac{1}{4} \frac{v_{\text{threshold}}}{d_0} = \gamma = \frac{\omega}{Q} \quad (34)$$

where $v_{\text{threshold}}$ is the amplitude of the second-harmonic component of the velocity of the moving mirror that is being driven by the second harmonic frequency 2ω of some external pump source, and where $\omega = \omega_s = \omega_i$ is the degenerate signal (or idler) frequency of the LC resonator. Solving for the velocity $v_{\text{threshold}}$ from (34), we find that the minimum velocity amplitude for the mirror necessary for the onset of degenerate parametric oscillation, is

$$v_{\text{threshold}} = \frac{4\omega d_0}{Q} \quad (35)$$

We can then calculate the minimum, time-averaged kinetic energy of the moving mirror caused by the pump at threshold as follows:

$$\langle K_{\text{threshold}} \rangle = \frac{1}{2} m \langle v_{\text{threshold}}^2 \rangle = \frac{1}{4} m \left(\frac{4\omega d_0}{Q} \right)^2 = \frac{4m\omega^2 d_0^2}{Q^2} \quad (36)$$

The total stored energy $U_{\text{threshold}}$ at threshold in the motion of the mirror is the sum of the threshold kinetic and potential energies, which are equal to each other for simple harmonic motion, so that

$$U_{\text{threshold}} = \langle K_{\text{threshold}} \rangle + \langle V_{\text{threshold}} \rangle = 2 \langle K_{\text{threshold}} \rangle \quad (37)$$

Therefore we conclude that the total pump energy required at threshold for exciting degenerate parametric oscillations of the “pellicle mirror” is

$$U_{\text{threshold}} = \frac{8m\omega^2 d_0^2}{Q^2} \quad (38)$$

where m is the mass of the “pellicle mirror”, ω is the degenerate signal (or idler) frequency of the generated microwaves in the “pillbox cavity” mode, and d_0 is the effective gap of the capacitance in the equivalent LC circuit for this cavity mode.

Let us compare this expression with the following expression for the threshold given by Braginsky et al. [5] for the generation of the parametric oscillations of an elastic mode of a moving mirror of a Fabry-Perot cavity excited by radiation pressure from a pump laser:

$$U_{\text{Braginsky's threshold}} = \frac{1}{2} \frac{m\omega_s^2 L^2}{Q_i Q_s} \quad (39)$$

where m is the mass of the moving mirror, ω_s is the signal frequency, L is the length of the Fabry-Perot cavity, and Q_i and Q_s are respectively the quality factors of the idler and signal modes of the parametric oscillator. We see that the expressions for the two thresholds (38) and (39) become identical, if one identifies

$$d_0 = \frac{1}{4}L \text{ and } Q_i = Q_s = Q \quad (40)$$

i.e., if the effective length scale of the capacitor d_0 in the LC model of the “pillbox cavity” in its TM 010 mode is identified with one quarter of the length L of a Fabry-Perot resonator in one of its TEM modes, and if the two quality factors $Q_i = Q_s$ become identical, since the signal and idler are being excited in the same, degenerate mode in the “pillbox cavity” case. In any case, the two thresholds (38) and (39) are comparable to each other in terms of orders of magnitude, which will be sufficient for designing experiments.

Going back to the original “pillbox” configuration of Figure 1(a), we estimate that the characteristic length scale for the “pillbox cavity” is

$$d_0 \simeq \lambda \quad (41)$$

which would be more appropriate for the TM 012 mode than the TM 010 mode. Now since

$$d_0\omega \simeq \lambda\omega \simeq 2\pi\lambda f \simeq 2\pi c \quad (42)$$

where λ is the wavelength of microwaves with a resonance frequency $f = \omega/2\pi$, and c is the speed of light, we conclude that a rough estimate of the threshold for degenerate parametric oscillation in (35) is given by the expression

$$\frac{v_{\text{pump}}}{c} \simeq \frac{2\pi}{Q} \quad (43)$$

where v_{pump} is the velocity of the moving mirror which is being driven at the second harmonic frequency by the pump.

Let us compare the condition (43) with the threshold condition introduced by Walls and Milburn [6] in connection with the production of squeezed states in parametric oscillation (as cited by Nation et al. [7])

$$\frac{\varepsilon\omega Q}{c} \geq 1 \quad (44)$$

where ε is the displacement amplitude of the moving mirror inside a Fabry-Perot resonator (with a fixed mirror and a moving mirror), whose resonance frequency is ω and whose quality factor is Q . The velocity amplitude of the sinusoidal motion of the mirror is given by

$$v_{\text{mirror}} = \varepsilon\omega \quad (45)$$

Therefore it follows that Walls and Millburn's threshold condition (44) can be re-expressed in terms of the mirror's maximum velocity v_{mirror} as follows:

$$\frac{v_{\text{mirror}}}{c} \geq \frac{1}{Q} \quad (46)$$

which is indeed consistent with (43), when viewed as an order-of-magnitude estimate of the threshold.

Since for superconducting cavities, it is possible to achieve the extremely high Q 's of the order of [8]

$$Q \simeq 10^{10} \quad (47)$$

the two threshold conditions (43) and (46) indicate that it is not necessary for the moving mirror to be driven into relativistic motion (i.e., with $v_{\text{mirror}} \simeq c$) by the pump source in order to achieve the threshold for parametric oscillation.

One practical way to drive the "pellicle mirror" into motion is to use a second high- Q microwave cavity to drive it mechanically via the Maxwell stress tensor on the other side of this mirror, so as to excite it at the second harmonic of the degenerate signal-idler cavity frequency (see Figure 3). For example, the "pillbox cavity" on the left side of the "pellicle mirror" could be this second cavity, which could be filled pump microwaves, whereas the "pillbox cavity" on the right side of the "pellicle mirror", which is initially filled only with the vacuum, could be filled with degenerate signal and idler microwaves upon exponential gain above threshold. The left cavity could be excited by means of an antenna consisting of an extension of the center conductor of the on-axis "IN" SMA cable. This cable could be charged with a DC bias potential

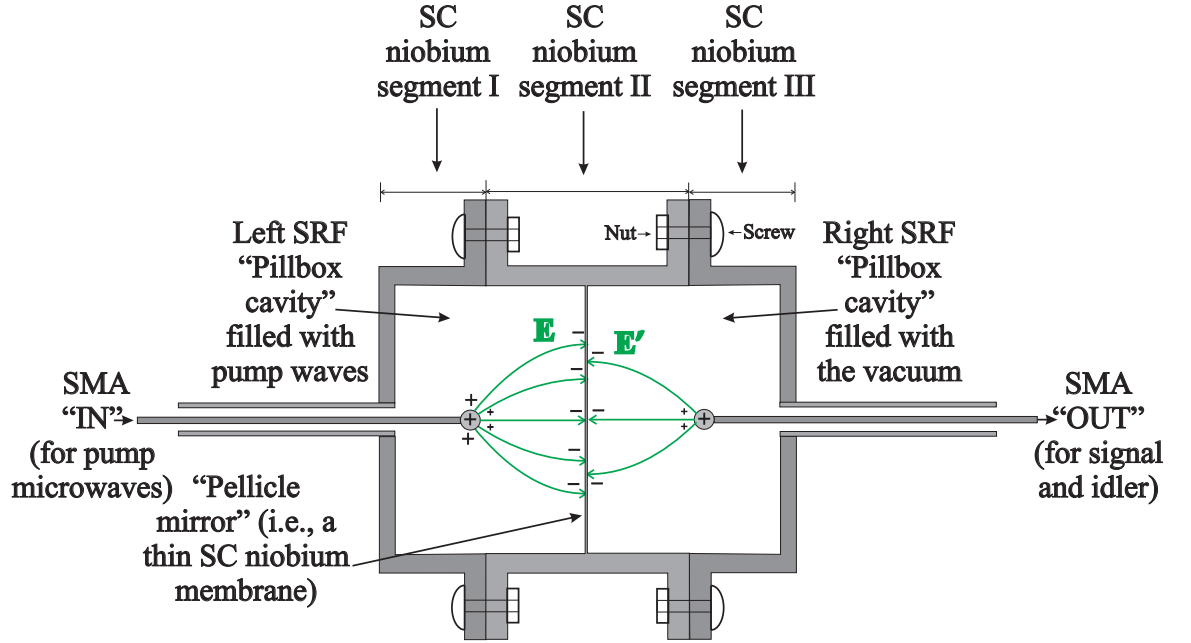


Figure 3: Degenerate parametric oscillator consisting of two superconducting radio frequency (SRF) “pillbox cavities” bolted together from three cylindrical segments (I, II, III) of superconducting (SC) niobium. The central segment (II) has a flexible “pellicle mirror” incorporated into its midsection as the active element of the amplifier. The “pellicle mirror” is a SC niobium membrane that serves as a thin wall that separates the entire assembly into left and right SC niobium cavities. The membrane can be charged on its two surfaces by electrostatic induction. The left cavity is pumped by microwaves injected into the “IN” port. Degenerate signal and idler microwaves grow exponentially out of vacuum fluctuations in the right cavity, and leave through the “OUT” port.

so as to induce electrostatic charges on the left inner surface of the “pellicle mirror”. Similarly, the radiation in the right cavity, which could be produced out of vacuum fluctuations in the dynamical Casimir effect, could be extracted by means of an output antenna which is the extension of the center conductor of the “OUT” SMA cable sticking out into this cavity. For the design of the nondegenerate parametric oscillator, see [10].

Let us calculate the threshold in terms of the external microwave pump power that needs to be injected through the “IN” port of the pump (i.e., left) cavity of Figure 3. To do so, we note that in steady-state equilibrium, the injected pump power coming in through the “IN” port at threshold must be balanced exactly by the lost pump power, i.e.,

$$\mathcal{P}_{\text{threshold}} = \frac{\langle U_{p,\text{threshold}} \rangle}{\tau_p} = \frac{\omega_p \langle U_{p,\text{threshold}} \rangle}{Q_p} \quad (48)$$

where $\tau_p = Q_p/\omega_p$ is the cavity ring-down time of the pump cavity.

In Figure 3, the external agency driving the “pellicle mirror” is the force associated with the Maxwell stress tensor, which gives rise to a pressure from the pump waveform that is being injected into the pump (left) cavity. Note that this force is exerted on the *opposite* side of “pellicle mirror” from the side that faces the signal (or idler) cavity, and is related to the acceleration of this “moving plate” in accordance with Newton’s 2nd law

$$F = PA = \left(\frac{1}{2} \varepsilon_0 E^2 \right) A = m \frac{d^2 x}{dt^2} \quad (49)$$

where $P = \frac{1}{2} \varepsilon_0 E^2$ is the pressure applied to the “plate,” A is the area of the “plate,” E is the total instantaneous electric field from the pump microwaves being applied at the “plate,” m is the mass of the “plate,” and x is the instantaneous displacement of the “plate.” To a good approximation, the moving mirror (i.e., the moving plate of the capacitor in Figure 2) is moving as if it were a *free* mass that is being driven by the pump microwaves.

First, we shall consider the case in Figure 3 in which there is no external DC bias injected through the “IN” port along with the microwave pump field. Let this pump wave have the waveform

$$E = E_p \sin \omega_p t \quad (50)$$

where E_p is the electric field amplitude of the pump wave, and the pump frequency ω_p is at the *first* harmonic (i.e., at the *same* frequency as the degenerate signal and idler frequencies $\omega_s = \omega_i$). However, note that the left and right cavities form two separate Faraday cages, so that there is no possibility of any leakage of pump radiation from the left cavity into the right cavity, which could be confused with the signal (or idler) radiation that is generated by the dynamical Casimir effect.

The square of E , which enters into (49), becomes

$$E^2 = E_p^2 \sin^2 \omega_p t = E_p^2 \left(\frac{1}{2} - \frac{1}{2} \sin 2\omega_p t \right) \quad (51)$$

which generates the second harmonic needed for pumping the degenerate parametric oscillations in the right cavity. Looking only the second harmonic component of Newton's equation of motion (49), we see that

$$F(\text{at } 2\omega_p) = \left(-\frac{1}{4}\varepsilon_0 E_p^2 \sin 2\omega_p t\right) A = m \frac{d^2 x}{dt^2} = -m(2\omega_p)^2 x \quad (52)$$

where the solution for the time-varying displacement is

$$x = x_p \sin 2\omega_p t \quad (53)$$

where, from (52), we find that the solution for the displacement amplitude is

$$x_p = \frac{1}{16} \frac{\varepsilon_0 E_p^2}{m\omega_p^2} \quad (54)$$

The solution for the time-varying velocity of the driven “pellicle mirror” is

$$v = \frac{dx}{dt} = v_p \sin 2\omega_p t \quad (55)$$

where the solution for the velocity amplitude is

$$v_p = \omega_p x_p = \frac{1}{8} \frac{\varepsilon_0 E_p^2}{m\omega_p} \quad (56)$$

Now from (38), we found that the threshold for the total mechanical energy of the driven “pellicle mirror” required to achieve degenerate parametric oscillation is

$$U_{\text{threshold}} = \frac{8m\omega_s^2 d_0^2}{Q_s^2} \quad (57)$$

where m is the mass of the “pellicle mirror”, $\omega = \omega_s$ is the signal (or idler) frequency, d_0 is the effective length in the LC model of the signal cavity, and $Q = Q_s$ is the quality factor of the signal (or idler) cavity (i.e., the right cavity of Figure 3).

In steady state, and in the absence of any dissipation into heat, we infer that the threshold total mechanical energy of the driven “pellicle mirror” will come into equilibrium with the time-averaged threshold microwave energy stored in the left (pump) cavity of Figure 3, i.e.,

$$U_{\text{threshold}} = \langle U_{p,\text{threshold}} \rangle \quad (58)$$

Therefore, in order to convert (57) into a threshold pump power, we shall use the steady-state relationship (48), which leads to the threshold condition [9]

$$\mathcal{P}_{\text{threshold}} = \frac{\omega_p U_{\text{threshold}}}{Q_p} = \frac{8m\omega_p\omega_s^2 d_0^2}{Q_s^2 Q_p} \quad (59)$$

This is the minimum amount of power needed to cause degenerate parametric oscillations of signal and idler waves that start to build up exponentially in the right cavity of Figure 3.

If we make the further simplifying assumption here that both the pump (left) and signal (right) cavities of Figure 3 are similar superconducting cavities, so that

$$Q_s = Q_p = Q \sim 10^{10} \quad (60)$$

then (59) simplifies to the expression

$$\mathcal{P}_{\text{threshold}} = \frac{8m\omega_p\omega_s^2d_0^2}{Q^3} \propto \frac{1}{Q^3} \quad (61)$$

i.e., the threshold pump power scales inversely as the *cube* of the Q of the superconducting cavities.

Let us now put in some numbers to see whether an experiment is feasible to do or not. Suppose that $m = 2$ milligrams (as in our FQMT11 paper [9]), and that $\omega_p \simeq \omega_s \simeq 2\pi \times 10^{10}$ Hz, and that $d_0 \simeq \lambda_s$, so that $\omega_s^2 d_0^2 = \omega_s^2 \lambda_s^2 \simeq 4\pi^2 c^2$. Then (61) becomes

$$\begin{aligned} \mathcal{P}_{\text{threshold}} &\simeq 32\pi^2 \cdot mc^2 \cdot \frac{\omega_p}{Q^3} \\ &\simeq 64\pi^3 \times 2 \times 10^{-6} \text{kg} \times \left(3 \times 10^8 \frac{\text{m}}{\text{s}}\right)^2 \times \frac{10^{10} \text{Hz}}{(10^{10})^3} \\ &\simeq 3.6 \text{ microwatts} \end{aligned} \quad (62)$$

which indicates that the experiment sketched in Figure 3 is indeed feasible to perform using our dilution refrigerators. The most critical and difficult part of the experiment will be to ensure that the Q of the superconducting cavities will indeed be on the order of 10^{10} or so.

Next, we shall consider the case in Figure 3 in which there exists a DC bias, so that a DC electric field E_0 is then superposed with a microwave pump field of amplitude E_p of a pump wave at the second harmonic of the signal. Therefore, let this pump have a waveform

$$E = E_0 - E_p \sin \omega_p t \quad (63)$$

where now the pump frequency ω_p is at the *second* harmonic (i.e., at *twice* the frequency as the degenerate signal and idler frequencies $\omega_s = \omega_i$). Then squaring this superposition of fields to find the force in (49), we obtain

$$E^2 = E_0^2 - 2E_0E_p \sin \omega_p t + E_p^2 \sin^2 \omega_p t \quad (64)$$

We shall adopt throughout this calculation the LC circuit model for the pump cavity, as well as for the signal cavity, in which all electric fields, like those in the capacitor C , are assumed to be uniform fields. If the DC electric field is much larger than the amplitude of the microwave pump waveform, i.e., if

$$E_0 \gg E_p \quad (65)$$

then the dominant time-varying term of (64) is the one at the first harmonic of ω_p , so that (49) becomes, to a good approximation,

$$-\left(\frac{1}{2}\varepsilon_0 \times 2E_0E_p \sin \omega_p t\right)A = m \frac{d^2x}{dt^2} \quad (66)$$

Then the solution for the displacement x of the plate is

$$x = x_p \sin \omega_p t \quad (67)$$

with a displacement amplitude

$$x_p = \frac{\varepsilon_0 E_0 E_p A}{m \omega_p^2} \quad (68)$$

Therefore the solution for the velocity v of the plate is

$$v = v_p \cos \omega_p t \quad (69)$$

with a velocity amplitude

$$v_p = \frac{\varepsilon_0 E_0 E_p A}{m \omega_p} \quad (70)$$

Identifying this velocity amplitude with the velocity amplitude in (28), we see that, for this kind of “external pumping agency,” i.e., the one using the pump in the left cavity of Figure 3 *with* a DC bias, the resulting driven velocity amplitude of the “pellicle mirror” is

$$v_{2\omega} = v_{\text{pump}} = v_p = \frac{\varepsilon_0 E_0 E_p A}{m \omega_p} \quad (71)$$

Solving for the pump electric field amplitude, one finds

$$E_p = \frac{m \omega_p v_p}{\varepsilon_0 E_0 A} \quad (72)$$

To calculate the threshold power in the presence of a DC bias for parametric oscillation of the degenerate signal and idler waves inside the right cavity, we shall assume that the left cavity of Figure 3, which is filled with pump waves, will fix the amplitude of the mechanical drive of the “pellicle mirror”, so that the pump velocity amplitude (71) remains unchanged (i.e., undepleted) by the generation of weak signal and idler waves in the right cavity. We shall then use the threshold condition (34) for the gain coefficient κ that we obtained earlier for the generation of degenerate signal and idler waves

$$\kappa_{p,\text{threshold}}^{(\text{DC bias})} = \frac{1}{4} \frac{v_{p,\text{threshold}}^{(\text{DC bias})}}{d_0} = \frac{\omega_s}{Q_s} = \frac{1}{2} \frac{\omega_p}{Q_s} \quad (73)$$

where Q_s is the Q factor of the right (signal) cavity of Figure 3. Solving for the threshold pump velocity, one finds

$$v_{p,\text{threshold}}^{(\text{DC bias})} = \frac{2d_0\omega_p}{Q_s} \quad (74)$$

Solving for the threshold electric field using (72) and (74), we find

$$E_{p,\text{threshold}}^{(\text{DC bias})} = \frac{2m\omega_p^2 d_0}{\varepsilon_0 E_0 A Q_s} \quad (75)$$

Identifying the time-averaged threshold electric field energy density in the pump cavity as

$$\langle u_E \rangle_{p,\text{threshold}}^{(\text{DC bias})} = \frac{1}{4} \varepsilon_0 \left(E_{p,\text{threshold}}^{(\text{DC bias})} \right)^2 \quad (76)$$

and expressing the total energy stored in the pump cavity, in the LC capacitor model for the cavity, as

$$\begin{aligned} \langle U_{\text{tot}} \rangle_{p,\text{threshold}}^{(\text{DC bias})} &= \left(\langle u_E \rangle_{p,\text{threshold}}^{(\text{DC bias})} + \langle u_B \rangle_{p,\text{threshold}}^{(\text{DC bias})} \right) A d_0 \\ &= \frac{1}{2} \varepsilon_0 \left(E_{p,\text{threshold}}^{(\text{DC bias})} \right)^2 A d_0 \end{aligned} \quad (77)$$

and using the fact that this energy is related to the threshold input power in steady state given by (48), we conclude from (75) that the threshold pump power in the presence of the DC bias is given by

$$\begin{aligned} \mathcal{P}_{\text{threshold}}^{(\text{DC bias})} &= \langle U_{\text{tot}} \rangle_{p,\text{threshold}}^{(\text{DC bias})} \cdot \left(\frac{\omega_p}{Q_p} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} \varepsilon_0 \left(E_{p,\text{threshold}}^{(\text{DC bias})} \right)^2 \right) A d_0 \cdot \left(\frac{\omega_p}{Q_p} \right) \\ &= \frac{1}{4} \varepsilon_0 \left(\frac{2m\omega_p^2 d_0}{\varepsilon_0 E_0 A Q_s} \right)^2 A d_0 \cdot \left(\frac{\omega_p}{Q_p} \right) \\ &= \frac{m^2 \omega_p^5 d_0^3}{\varepsilon_0 E_0^2 A Q_s^2 Q_p} \end{aligned} \quad (78)$$

If we again make the simplifying assumption that

$$Q_p = Q_s = Q \quad (79)$$

because both the left and right cavities are superconducting cavities with similar configurations, then (78) becomes

$$\mathcal{P}_{\text{threshold}}^{(\text{DC bias})} = \frac{m^2 \omega_p^5 d_0^3}{\varepsilon_0 E_0^2 A Q^3} \propto \frac{1}{Q^3} \quad (80)$$

i.e., the threshold once again depends on the inverse *cube* of the quality factor of the superconducting cavities. Since for the TM 011 mode, the distance scale for the capacitor gap is approximately

$$d_0 \simeq \frac{\lambda_p}{2} = \frac{\pi c}{\omega_p} \quad (81)$$

the factor $\omega_p^5 d_0^3$ simplifies to become

$$\omega_p^5 d_0^3 \simeq \omega_p^2 \pi^3 c^3 \quad (82)$$

Hence the threshold pump power simplifies to become

$$\mathcal{P}_{\text{threshold}}^{(\text{DC bias})} \simeq \pi^3 \frac{m^2 \omega_p^2 c^3}{\varepsilon_0 E_0^2 A Q^3} = \pi^3 \frac{m^2 c^4 \omega_p^2}{\varepsilon_0 E_0^2 A c Q^3} \quad (83)$$

If we now take the ratio of the expression with the DC bias (83) with the expression for the case without the DC bias (62), we see that

$$\begin{aligned} \frac{\mathcal{P}_{\text{threshold}}^{(\text{DC bias})}}{\mathcal{P}_{\text{threshold}}} &\simeq \left(\pi^3 \frac{m^2 c^4 \omega_p^2}{\varepsilon_0 E_0^2 A c Q^3} \right) \cdot \left(32 \pi^2 \cdot m c^2 \cdot \frac{\omega_p}{Q^3} \right)^{-1} \\ &= \frac{\pi}{32 \varepsilon_0 E_0^2 A (c/\omega_p)} = \frac{\pi^2}{32 \varepsilon_0 E_0^2 A d_0} \\ &= \frac{\pi^2}{32 \varepsilon_0 E_0^2 V_0} \end{aligned} \quad (84)$$

where $V_0 = A d_0$ is the volume of the pump cavity. It is obviously the case that the rest mass of the “pellicle mirror” is much larger by many orders of magnitude than the energy stored in the DC electric field inside the pump cavity, i.e.,

$$m c^2 \gg \varepsilon_0 E_0^2 V_0 \quad (85)$$

Hence we definitely should *not* use the DC bias method for trying to achieve the threshold of the degenerate parametric oscillator. Nevertheless, the use of an adjustable DC bias as indicated in Figure 3 could be useful as an adjustable experimental knob for tuning the two microwave cavities into resonance with respect to each other.

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References

- [1] One can derive (5) in the following way: An instantaneous force dF on a given capacitor plate arises from adding an instantaneous charge dQ to this plate in the presence of the instantaneous electric field E inside the capacitor through the relationship

$$dF = dQ \cdot E \quad (86)$$

However, by Gauss's law applied to a pillbox staddling the inner surface of the fixed left plate, one finds that the instantaneous electric field in between the capacitor plates is given by

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \quad (87)$$

where $\sigma = Q/A$ is the charge density on the plate of area A . From (87), it follows that

$$dQ = dE \cdot (\varepsilon_0 A) \quad (88)$$

Therefore the incremental force dF on the plate upon the increment of the charge dQ becomes

$$dF = dQ \cdot E = E \cdot dE \cdot (\varepsilon_0 A) \quad (89)$$

By integration, one finds that the total force is given by

$$F = (\varepsilon_0 A) \int_0^E E dE = \left(\frac{1}{2} \varepsilon_0 E^2 \right) A \quad (90)$$

and therefore that the pressure (i.e., a component of the Maxwell stress tensor) is given by

$$P = \frac{F}{A} = \frac{1}{2} \varepsilon_0 E^2 = u_E \quad (91)$$

where u_E is the instantaneous energy density stored in the electric field inside the capacitor. From this derivation, it is clear that it is the *Coulomb* force, and not the *Lorentz* $\mathbf{v} \times \mathbf{B}$ force associated with radiation pressure, which is operative in driving the motion of the moving mirror in the TM mode pictured in Figure 1. This implies that the Maxwell stress tensor (6) should be much more effective than radiation pressure in producing parametric gain and oscillation of a moving mirror. Now consider the lowest possible transverse magnetic mode, i.e., the TM 010 mode, which is a mode with longitudinal electric field lines that terminate on the charges at the end faces of the cavity, as indicated in Figure 1. These charges are necessary for there to exist a longitudinal force associated with the Maxwell stress tensor, which acts along the axis of the pellicle mirror to drive it into motion. By contrast, the lowest possible transverse electric mode, i.e., the TE 111 mode, has transverse, but no longitudinal, electric field lines. Hence no longitudinal force can arise from the charges induced by this TE mode. However, there does exist a longitudinal radiation pressure force arising from the Poynting vector at the surface of the "pellicle mirror" for the TE modes, but it is much smaller in magnitude than the longitudinal force arising from the Maxwell stress tensor for the TM modes.

[2] The sign of the work as defined here in (1) and (4) is opposite to the sign of the work as defined in thermodynamics, which is the work done

by a piston *on* an external agency. In thermodynamics, by convention, the sign of the work is *positive* when the work is done *by* the “system” *on* the “environment,” i.e., when useful thermodynamic work can be extracted *from* the “system” *via* the piston *into* the “environment.” Here, by contrast, the sign of the mechanical work in (1) and (4) is *positive* when the work is done *by* the external agency (i.e., the pump) *on* the capacitor plate (i.e., the fields of the LC circuit), so that useful parametric *amplification* can occur.

- [3] Note that from (1) and (4), we could interpret the action of the external pump agency upon the “pellicle mirror” as effectively that of a “piston” performing *negative* thermodynamic PdV work, i.e., *positive* mechanical work *by* the external agency *on* an effective “gas” of “virtual photons” (i.e., vacuum fluctuations), which are stored in the form of the fluctuating electric field inside the volume of the capacitor C in Figure 2, although this quantum interpretation can be safely ignored in the above classical discussion.
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- [9] The expression (59) is consistent with the expression for the threshold power given in Equation (10) of our paper in FQMT11 “Parametric Oscillation of a Moving Mirror Driven by Radiation Pressure in a Superconducting Fabry-Perot Resonator System,” which was based on Braginsky’s threshold condition (39) (to be published in *Physica Scripta*). See arXiv: 1207.6885.
- [10] For the nondegenerate case, a “double cavity” with an iris in its middle could be used for splitting the resonance for the signal and idler frequencies into a close doublet, and then coupled to a high- Q pump cavity to form a “triple cavity,” as sketched in Figure 4.

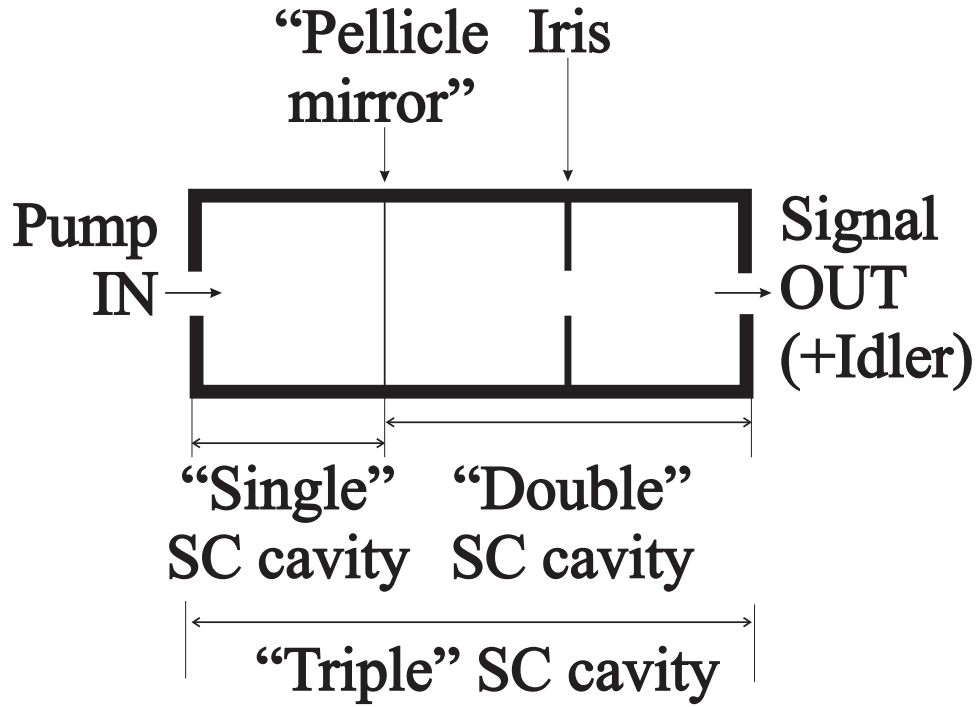


Figure 4: Simplified schematic of a “triple SC cavity” consisting of a combination of a “single SC cavity” for the pump wave, and a “double SC cavity” for *nondegenerate* signal and idler waves. A thin, flexible “pellicle mirror” separates the “single” from the “double” cavity, and is driven into motion at microwave frequencies by the pump wave. The iris placed at the midpoint of the “double” cavity couples its two halves together, so that there results a doublet of two closely spaced mode frequencies resonant with the nondegenerate signal and idler waves.